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Stationary Solutions of an Equation Modelling Ohmic Heating

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Abstract—In this note, we study questions of multiplicity and stability of stationary solutions of the nonlocal reaction-diffusion equation $u_t = u_{xx} + \lambda f(u)/\left(a + \int_0^1 f(u)\,dx\right)^2$ which arises in the theory of electrical devices with temperature-dependent resistivity and where $f(u)$, which is taken to be a strictly positive function, represents the temperature-dependent resistivity. We also prove that solutions exist for all positive time and must enter a bounded region as t goes to infinity.

Keywords—Ohmic heating, Nonlocal parabolic equations.

1. INTRODUCTION

In this note, we consider the stationary solutions of the nonlocal reaction-diffusion equation

$$u_t = u_{xx} + \lambda \frac{f(u)}{\left[a + \int_0^1 f(u)\,dx\right]^2} \tag{1}$$

with the homogeneous Dirichlet conditions

$$u(0,\ t) = u(1,\ t) = 0.$$

This equation arises as a model for the temperature in devices such as the thermistor, where an electric current flows through a material with temperature-dependent electrical resistivity. Here, it is assumed that the device is part of a circuit in series with a constant resistor R and driven by a constant voltage E. Then, if A denotes the area of the cross-section of the arc, $\lambda = E^2$ and $a = RA$. For more details, including an outline of the model, see [1,2].

The purpose of the present analysis is to study the effect of the additional resistor and to see to what extent the results obtained in [1] are changed. In particular, we will show that it is now possible to have multiple stationary solutions when a is large enough, while it was shown in [1] that when $a = 0$, there will always exist one and only one stationary solution. In addition, we consider stability of stationary solutions and discuss both increasing and decreasing resistivity

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functions $f(u)$. We use the local time-map, a differential equation for which is derived using Picard-Fuchs equations, to analyse the nonlocal case. For a more elementary approach, see [1,3] though other first-order differential equation formulations are available, only the Picard-Fuchs approach seems to give the time-map for the nonlocal problem at all easily and moreover, this approach can be applied to other types of nonlocal boundary value problems (see [4,5] for an application to boundary value problems with an integral constraint).

2. GLOBAL EXISTENCE

It has already been shown in [1] that solutions of (1) exist globally and are bounded. However, the bound obtained there depends on the initial condition. Here, we prove that they must enter a bounded region which depends only on λ, a and f, as t goes to infinity.

LEMMA 1.

$$\limsup_{t\to\infty} u(x,t) \le \frac{\lambda}{3(a+b)},$$

where $b = \inf\,[f(u)]$.

PROOF. Write u as a Fourier series with time dependent coefficients, that is,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x),$$

where $u_n(t) = 2\int_0^1 u(x,t)\sin(n\pi x)dx$. Then, multiplying both sides of (1) by $\sin(n\pi x)$ and integrating from 0 to 1 gives

$$u_n'(t) + n^2\pi^2 u_n(t) = 2\lambda\frac{\int_0^1 f(u)\sin(n\pi x)dx}{\left[a+\int_0^1 f(u)dx\right]^2},$$

which can be integrated to obtain

$$u_n(t) = e^{-n^2\pi^2 t}u_n(0) + 2\lambda\int_0^t \frac{e^{n^2\pi^2(s-t)}\int_0^1 f(u(x,s))\sin(n\pi x)\,dx}{\left[a+\int_0^1 f(u(x,s))\,dx\right]^2}\,ds.$$

Taking absolute values on both sides yields

$$|u_n(t)| \le |u_n(0)|e^{-n^2\pi^2 t} + \frac{2\lambda}{n^2\pi^2(a+b)}\left(1-e^{-n^2\pi^2 t}\right),$$

as

$$\frac{1}{\left[a+\int_0^1 f(u)dx\right]^2} \le \frac{1}{(a+b)\int_0^1 f(u)\,dx},$$

where $b = \inf\,[f(u)]$. This implies that

$$\limsup_{t\to\infty} |u_n(t)| \le \frac{2\lambda}{n^2\pi^2(a+b)}$$

and, from the expansion of u, we finally have

$$\limsup_{t\to\infty} u(x,t) \le \limsup_{t\to\infty}\sum_{n=1}^{\infty}|u_n(t)| \le 2\lambda\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2(a+b)} = \frac{\lambda}{3(a+b)},$$

as the series of $|u_n(t)|$ is uniformly convergent. ∎

Clearly, the argument still holds if $a = 0$, provided that f is bounded away from zero.

3. EXISTENCE OF STATIONARY SOLUTIONS

To understand the structure of the stationary solutions of the problem (1), we consider first the problem

$$0 = u_{xx} + \mu f(u), \tag{2}$$

where $f(u) > 0$ is a Lipschitz continuous function, with the Dirichlet boundary conditions

$$u(0) = u(1) = 0.$$

The structure of solutions of this problem is well known (see e.g., [6,7]). To analyse the nonlocal problem, we shall use the time-map for this equation. In some special cases (for example, if $f(u) = e^u$ or $f(u) = e^{-u}$), it can be constructed explicitly by solving the relevant (linear in that instance!) Picard-Fuchs type equation.

Set $\mu = L^2$, $y = Lx$, $z(y) = u(Lx)$. Then, z satisfies the equation

$$z_{yy} + f(z) = 0, \quad y \in (0,\, L), \quad z(0) = z(L) = 0.$$

Considering this as an initial value problem, we see that it is Hamiltonian with the Hamiltonian

$$H(z,\, z_y) = \frac{1}{2} z_y^2 + F(z),$$

with the potential $F(z) = \int^z f(v)dv$. In the $(z,\, z_y)$ plane, consider a point $(0,\, q)$, $q > 0$. Then, a solution of the Dirichlet problem (for some value of L) is a piece of the level curve of H from $(0,\, q)$ to $(0,\, -q)$. Clearly, the phase portrait is symmetric with respect to the z-axis. Note that under the assumption of positivity of f, $F(z)$ is a monotone increasing function. Denote by $\bar{z}(q)$ the point of intersection of the level curve through $(0,\, q)$ with the z-axis; let $p = F(\bar{z}(q))$. Call Γ the piece of the level curve between $(0,\, q)$ and $(\bar{z}(q),\, 0)$. Finally, let $v^2(p,\, z) = 2[p - F(z)]$. As differential forms, $dy = dz/v$.

Note that

$$\int_0^L dy = L = 2\int_0^{L/2} dy.$$

Therefore, if we consider L as a function of p (this is the time-map), we have

$$L(p) = 2\int_\Gamma \frac{dz}{v}.$$

Now

$$\int_\Gamma v\, dz = \int_\Gamma \frac{v^2 dz}{v} = 2p\int_\Gamma \frac{dz}{v} - 2\int_\Gamma \frac{F(z)dz}{v}.$$

Therefore, if we use the notation

$$\eta(p) = \int_\Gamma \frac{F(z)dz}{v},$$

we see that $L(p)$ satisfies the differential equation

$$p\frac{dL}{dp} + \frac{L}{2} - 2\eta'(p) = 0 \tag{3}$$

for $p \in (F(0),\, F(\infty))$, with the initial condition $L(F(0)) = 0$. Obviously, the properties of $L(p)$ for large p depend crucially on the function $\eta(p)$. In certain cases, this equation can be solved explicitly (see below).

Clearly, solutions of the local problem are in one-to-one correspondence with solutions of the equation $L(p) = \sqrt{\mu}$.

To understand the nonlocal problem, we need the function

$$\nu(p) = \int_0^1 f(u(x))\,dx,$$

that is, given p, $\nu(p)$ gives us the integral of the solution corresponding to that p.

Clearly,

$$\nu(p) = \frac{2}{L(p)} \int_0^{L(p)/2} f(z)\,dy = \frac{2}{L(p)} \int_\Gamma \frac{f(z)dz}{v} = \frac{2^{3/2}}{L(p)}(p - F(0))^{1/2}.$$

LEMMA 2. *The solutions of the nonlocal problem are in one-to-one correspondence to the solutions of the equation*

$$\pi(p) = \left(aL(p) + 2^{3/2}(p - F(0))^{1/2}\right) = \sqrt{\mu}. \tag{4}$$

PROOF. For every $p \in (F(0), F(\infty))$, there exists a unique pair $(u_p(x),\ \mu_p)$, $\mu_p > 0$ which satisfies the local problem:

$$(u_p)_{xx} + \mu_p f(u_p) = 0$$

on $x \in (0\ 1)$ with Dirichlet boundary conditions. In fact, $u_p(1/2) = F^{-1}(p)$, and this follows by monotonicity of $F(u)$ and uniqueness of solutions of the respective initial value problem.

Furthermore, in terms of the time-map $L(p)$, $\sqrt{\mu_p} = L(p)$. Define now

$$\lambda_p = \mu_p \left[a + 2^{3/2} L(p)(p - F(0))^{1/2}\right]^2.$$

Then, $(u_p,\ \lambda_p)$ also solves the nonlocal problem, that is,

$$(u_p)_{xx} + \frac{\lambda_p f(u_p)}{\left[a + \int_0^1 f(u_p)\,dx\right]^2} = 0.$$

Therefore, to each $p \in (F(0),\ F(\infty))$ corresponds a unique pair $\left(u_p,\ \overline{\lambda_p}\right)$, which solves the nonlocal problem.

In fact, in this fashion we obtain all the solutions of the nonlocal problem. For consider a solution pair $(u,\ \lambda)$ of the nonlocal problem. It defines a unique solution pair $\left(u,\ \lambda / \left[a + \int_0^1 f(u)\right]^2\right)$ of the local problem, which in turn uniquely defines p by $p = F(u(1/2))$. But then, by the above argument we must have $(u,\ \lambda) = (u_p,\ \lambda_p)$. ∎

$\pi(p)$ is the time-map for the nonlocal problem. From (4) and the results of [7] for the local problem, we obtain

THEOREM 1. *For all $a > 0$, there exists a positive stationary solution of (1) for all values of $\lambda > 0$.*

REMARK. In the case $a = 0$, we can rederive from (4) the following result from [1]:

THEOREM 2. *If $a = 0$, then:*

(1) *If $\int_0^\infty f(s)\,ds = \infty$, then there exists a unique positive stationary solution of (1) for all $\lambda > 0$.*
(2) *If $\int_0^\infty f(s)\,ds < \infty$, then there exists a unique positive stationary solution of (1) for all $\lambda < \lambda_* = 8\int_0^\infty f(s)\,ds$ and there are none for $\lambda \geq \lambda_*$. Furthermore, $\lim_{\lambda \to \lambda_*} \sup_x u(x) = \infty$.*

4. EXACT MULTIPLICITY RESULTS

When more information is available concerning $L(p)$, we can prove exact multiplicity results for the nonlocal problem. For example, when $L(p)$ is monotone increasing, stationary solutions for $a > 0$ are always unique. This includes, for instance, the cases where f is monotone decreasing or sublinear and concave.

We shall now consider two cases.

4.1. $f(u) = e^u$.

It is easy to check that in this case (3) is

$$p\frac{dL}{dp} + \frac{L}{2} - 2^{1/2}(p-1)^{-1/2} = 0$$

for $p > 1$ subject to $L(1) = 0$. This can be solved to give

$$L(p) = \sqrt{\frac{2}{p}} \ln\left(2p - 1 + 2\sqrt{p(p-1)}\right).$$

As everything here is explicit, we have the following statement.

LEMMA 3. *If $a = 0$, $\sup_x u(x) = \ln(\lambda/8 + 1)$. There exists $a_* > 0$, such that for $a \le a_*$, (1) has a unique positive stationary solution, while for $a > a_*$ there exist numbers $0 < \lambda_1 < \lambda_2$, such that if $\lambda_1 < \lambda < \lambda_2$, it has precisely three positive stationary solutions and only one otherwise.*

Numerically, $a_* \approx 11.5$. The result means that, in this case, incorporating a sufficiently strong resistor in series with our device would lead to a jump in temperature as the potential difference is slowly increased.

REMARK. Multiple steady states will always occur if $f(u)$ has superlinear growth and a is large enough: from results of [7], it follows that, in that case, $\lim_{p\to\infty} L(p) = 0$, and the result is then obtained by considering (4).

4.2. $f(u) = e^{-u}$.

In this case also, (3) can be solved ($p \in (-1, 0)$) to give

$$L(p) = 2^{3/2}\frac{\frac{\pi}{2} - \arcsin\sqrt{-p}}{\sqrt{-p}}.$$

LEMMA 4. *If $a = 0$, $\sup_x u(x) = -\ln(1 - \lambda/8)$, $\lambda < 8$. For $a > 0$, there exists a unique positive stationary solution of (1).*

5. STABILITY IN THE CASE $f(u) = e^u$

Concerning stability of stationary solutions in this case, we have the following lemma.

LEMMA 5. *If for $f(u) = e^u$ at a solution of $\pi(p) = \sqrt{\lambda}$ we have $\pi'(p) < 0$, then the corresponding solution of (1) is linearly unstable with a one-dimensional unstable manifold. If $\pi'(p) > 0$, it is linearly stable.*

PROOF. The linearised operator around a stationary solution w is defined by

$$L_1 u = u'' + \frac{\lambda e^w}{\left[a + \int_0^1 e^w dx\right]^2} u - \frac{2\lambda e^w}{\left[a + \int_0^1 e^w\, dx\right]^3} \int_0^1 e^w u\, dx.$$

Proceeding as in [8], consider the family of linear operators defined by

$$L_\epsilon u = u'' + \frac{\lambda e^w}{\left[a + \int_0^1 e^w \, dx\right]^2} u - \epsilon \frac{2\lambda e^w}{\left[a + \int_0^1 e^w \, dx\right]^3} \int_0 e^w u \, dx,$$

for real ϵ. Then, the spectrum of L_ϵ consists only of isolated eigenvalues with finite multiplicities. Also, as L_ϵ is, in this case, self-adjoint, these eigenvalues will be real. It is now possible to look at L_ϵ as a linear perturbation of the linear operator A corresponding to the linearisation of the standard local problem, that is

$$Av = v'' + \frac{\lambda e^w}{\left[a + \int_0^1 e^w dx\right]^2} v.$$

In this case, it is known that the eigenvalues form an ordered sequence $\gamma_0 > \gamma_1 > \cdots$ and that $\gamma_k \to -\infty$ as $k \to \infty$. Also, there is at most one positive eigenvalue. Using now the results from [8], it is possible to prove that, as ϵ goes from 0 to 1, the eigenvalues of L_ϵ will either move to the left or remain identical to γ_k. Hence, L_1 can have at most one positive eigenvalue. In the case where $\|w\|_\infty$ is small, w is a stable solution of the local problem and so $0 > \gamma_0 > \gamma_1 > \cdots$. This implies that the eigenvalues $\sigma_k(\epsilon)$ of L_ϵ are also negative as $\sigma_k(\epsilon) \leq \gamma_k$ for positive ϵ. In the case of $a < a_*$, the stationary solution remains isolated and is, therefore, stable for all λ. For $a > a_*$, Lemma 3 implies that there are two turning points in the bifurcation diagram. Using the same argument as before, we have that w must remain stable until the first turning point is reached, where there is an eigenvalue crossing zero and becoming positive. Hence, w becomes unstable with a 1-dimensional unstable manifold and it must remain like this until it reaches the second turning point. Here, again an eigenvalue crosses zero and, as there can exist at most one positive eigenvalue, we have that w must become stable again. ∎

REMARK. Note that this is the nonlocal equivalent of a result in [9]. In this particular case of an exponential nonlinearity, Lemma 5 can be also proved by observing that

$$\mathcal{L}(u) = \frac{1}{2} \int_0^1 u_x^2 \, dx + \frac{\lambda}{\left[a + \int_0^1 e^u \, dx\right]}$$

is a Liapunov function for (1). Lemma 1 provides us with an invariant interval in $L^\infty(0, 1)$. One can then apply the Conley index theory precisely in the same way it is done for the local scalar reaction-diffusion equation in [10].

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